A Proposed Nomenclature for Scalar Algebras

Abstract

Much of modern algebra is characterized by 16 axioms. These axioms are typically defined with respect six common operations on a class *C*. The class *C* is usually, but not always, a set. In order to keep the discussion simple, we will restrict ourselves to algebras of scalars.

The six operations

Addition (+) and multiplication (·) are binary operations on *C*. The additive (inv_+) and multiplicative ($inv.$) inverses are unary operations on *C.* The additive (Id_+) and multiplicative ($Id.$) identities are elements of *C*. They can be views as 0-ary operations on *C*.

The 16 Axioms + Three Additional Axioms

0(a) $a + b = b + a$
0(b) $a \cdot b = b \cdot a$
0(c) $inv_+(a) = inv.(a)$

1(a) $(a + b) + c = a + (b + c)$
1(b) $Id_+ + a = a + Id_+ = a$
1(c) $a + inv_+(a) = inv_+(a) + a = Id_+$
1(d) $a + inv_+(a) = inv_+(a) + a = Id.$

2(a) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2(b) $Id. \cdot a = a \cdot Id. = a$
2(c) $a \cdot inv.(a) = inv.(a) \cdot a = Id.$
2(d) $a \cdot inv.(a) = inv.(a) \cdot a = Id_+$

3(a) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
3(b) $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$
3(c) $a + (b \cdot c) = (a + b) \cdot (a + c)$
3(d) $(a \cdot b) + c = (a + c) \cdot (b + c)$

4(a) $Id_+ \cdot a = Id_+$
4(b) $a \cdot Id_+ = Id_+$
4(c) $Id. + a = Id.$
4(d) $a + Id. = Id.$

For an axiom to hold for a particular algebra, it must hold for every *a*, *b*, and *c* in the class *C*. Axioms 3(d), 4(b), and 4(d), represented in light blue font are not in common use. They are included in the list to allow for the construction new algebras by commutation or by interchanging multiplication and addition.

I will refer to an algebra that is defined solely by the six operations and the 16+3 axioms listed above as a **scalar algebra**. Since there are only 19 permitted axioms for any given scalar algebra, there are at most 2^{19} permitted scalar algebras. The axioms have been grouped into five categories numbered 0 through 4. Since there are at most four axioms in each category, the type of algebra may be described using a five digit hexadecimal number.

The digit for each category is assigned an odd digit if and only if the (a) axiom in that category holds. The (b) axiom holds if and only if the division of the digit by two yields an odd number, ignoring the remainder. The (c) axiom holds if and only if the division of the digit by four yields an odd number, ignoring the remainder. The (d) axiom holds if and only if the division of the digit by eight yields an odd number, ignoring the remainder.

The category digits are sequenced from right to left as follows: $D_4D_3D_2D_1D_0$ where D_n denotes the digit corresponding to category *n*. These rules completely define scalar algebra types.

Summary of Nomenclature

Name of Algebra	Axioms of the Algebra																Type of Algebra
Magma																	00000
Semigroup								2(a)									00100
Monoid								2(a)	2(b)								00200
Group								2(a)	2(b)	2(c)							00700
Abelian magma	0(a)																00001
Abelian semigoup	0(a)			1(a)													00011
Abelian monoid	0(a)			1(a)	1(b)												00031
Abelian group	0(a)			1(a)	1(b)	1(c)											00071
Rng	0(a)			1(a)	1(b)	1(c)		2(a)				3(a)	3(b)				03171
Ring	0(a)			1(a)	1(b)	1(c)		2(a)	2(b)			3(a)	3(b)				03371
Commutative rng	0(a)	0(b)		1(a)	1(b)	1(c)		2(a)				3(a)	3(b)				03173
Commutative ring	0(a)	0(b)		1(a)	1(b)	1(c)		2(a)	2(b)			3(a)	3(b)				03373
Field	0(a)	0(b)		1(a)	1(b)	1(c)		2(a)	2(b)	2(c)		3(a)	3(b)				03773
Near-semiring				1(a)	1(b)			2(a)							4(a)		10130
Nearring				1(a)	1(b)	1(c)		2(a)					3(b)				02170
Semiring	0(a)			1(a)	1(b)			2(a)	2(b)			3(a)	3(b)		4(a)		13331
Boolean algebra	0(a)	0(b)	0(c)	1(a)			1(d)	2(a)			2(d)	3(a)		3(c)	4(a)	4(c)	35997